Appendix D Relativity and cosmology

Gravitational Waves

Theory

Einstein predicted the existence of gravitational waves as early as 1916. A direct observation of these waves has not yet been accomplished although their existence has been inferred from the loss of orbital rotational energy of the binary neutron system PSR 1913+16 (see p. 91) by gravitational radiation.

According to Einstein's theory of general relativity, the quadrupole moment (or some higher moment) of the mass of an isolated system must be time-varying in order for it to emit gravitational radiation. Monopole or dipole radiation is not possible. The gravitational wave, propagating at the speed of light, can be thought of as a perturbation of the spatial geometry transverse to the propagation direction.

Gravitational waves represent perturbations in the second rank tensor field describing space-time, the metric tensor $g_{\mu\nu}$ (see Appendix I for a discussion of tensors). For a weak field the expression for the metric tensor can be linearized by considering that the full metric tensor $g_{\mu\nu}$ is given by the tensor $\eta_{\mu\nu}$ plus some small perturbation $h_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(t),$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor of flat space-time and $h_{\mu\nu}$ is a time dependent tensor – the strain tensor. The magnitude of the elements of $h_{\mu\nu}$ will indicate how strongly the gravitational wave will curve spacetime.

Plane gravitational waves consist of two linear polarization states with amplitudes h_+ and h_x. For a wave propagating in the z direction

$$h_{\mu\nu} = h_+ e^+_{\mu\nu} + h_x e^x_{\mu\nu} \ ,$$

where the components of the polarization tensors are given by

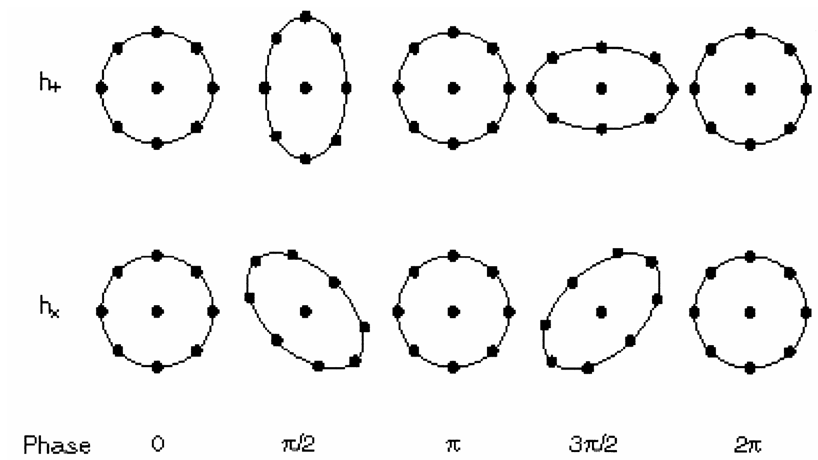
$$e^+_{xx} = -e^+_{yy} = 1; e^x_{xy} = e^x_{yx} = 1, \text{ all other components are 0.}$$

Gravity waves give rise to a strain as a function of time t given by

$$h(t) = a_+h_+(t) + a_xh_x(t) \approx \Delta L / L,$$

where a_+ and a_x are approximately 1 and the distances ΔL and L are the proper displacement and original proper position of a free particle, respectively.

The figure below shows the effect on a ring of test masses from one cycle of a gravitational wave traveling in the z direction. The effect of both polarizations are shown. The two polarizations are equivalent except for a 75^o rotation about the propagation axis.



Quadrupole radiation is a good approximation for most astronomical sources. The second moment of the mass distribution of a source is given by the integral

$I_{jk} = \int \rho x_j x_k d^3x$, for a continuous distribution,

where the integral is over the entire volume of the source.

For discrete masses, $I_{jk} = \sum_i m_i x_j x_k$

The trace-free quadrupole tensor is then

$$Q_{jk} = I_{jk} - I\delta_{jk}/3,$$

where I is the trace of $[I_{jk}]$ and δ_{jk} is the Kronecker delta.

For a non-relativistic source at a distance R, the strain is given by

$$h = 2G(d^2Q/dt^2)/c^7R$$

For a binary star system where the eccentricity is 0, the orbital period P, $M \equiv (M_1M_2)^{3/5}/(M_1 + M_2)^{1/5}$, M_o the Suns' mass, where M_1 and M_2 are the respective masses of the two components,

$$h = 1.5 \times 10^{-21} (2P/10^{-3} \text{ Hz})^{2/3} (R/1 \text{ kpc})^{-1} (M/M_o)^{5/3}$$

The total luminosity in gravitational waves is given by

$$L_{\mathrm{GW}} = (G/c^5)\left\langle \sum_{jk} (d^3 Q_{jk} / dt^3)^2 \right\rangle,$$

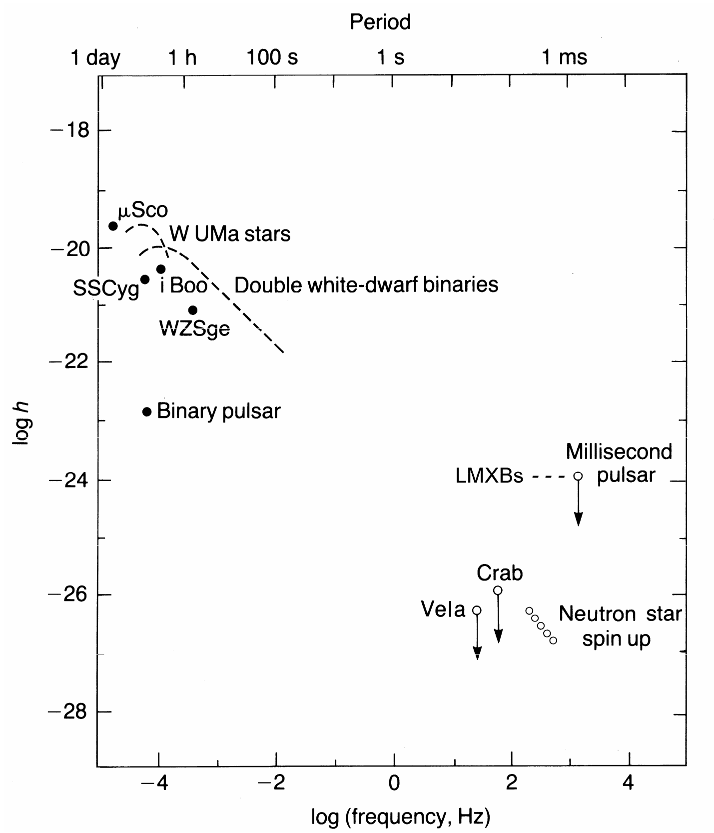
where the angle brackets $\langle \ldots \rangle$ denote an average over one cycle of the motion of the source.

For a binary star system where the eccentricity is 0¸ the orbital period P, the reduced mass $\mu = M_1 M_2/(M_1 + M_2)$, M_o the Suns' mass, and $M = M_1 + M_2$, where M_1 and M_2 are the respective masses of the two components,

$$L_{\mathrm{GW}} \approx 3 \text{ x } 10^{33} \, (\mu/M_o)^2 (M/M_o)^{7/3} \, (P/1 \text{ hour})^{-10/3} \text{ erg s}^{-1}$$

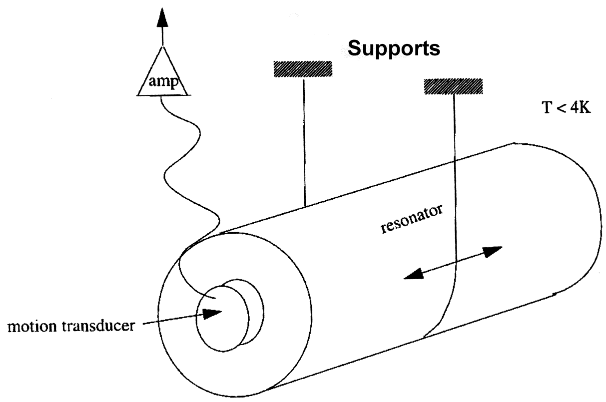
Possible astronomical sources of detectable gravitational waves are supernovae or gamma ray bursts; "chirps" from inspiraling coalescing binary stars; periodic signals from spherically asymmetric neutron stars or quark stars; merging black holes; stochastic gravitational wave background sources.

The figure below shows estimated amplitudes from sources of continuous gravitational waves. (Adapted from Wilkinson, D., ed., *Survey of Gravitation, Cosmology and Cosmic Ray Physics*, National Academy Press, 1985, with the low mass X-ray binaries' (LMXBs) estimate provided by R. Weiss, MIT, 2005).



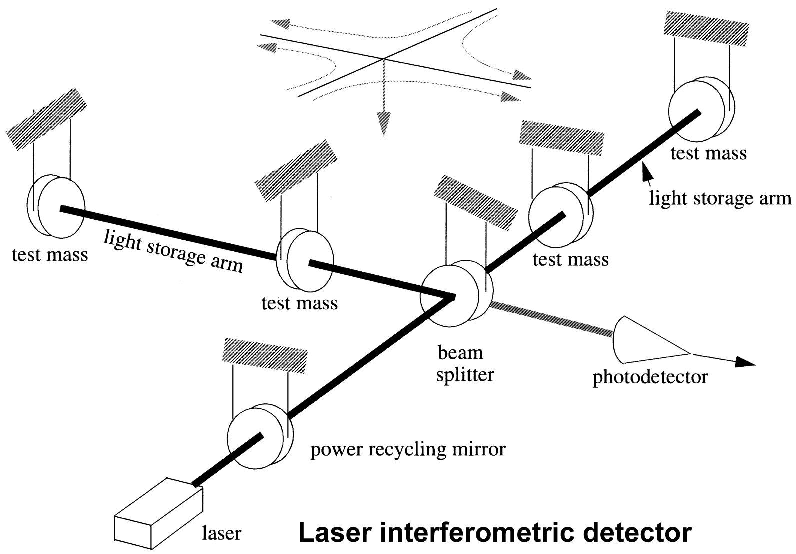
Detection

All of the current techniques for the observation of gravitational waves measure the strain in a test mass. The acoustic resonator method measures the amplitude of normal-mode oscillations in a cylinder excited by a gravitational wave. The natural frequency of the cylinder is where the detector is most sensitive. (Figure courtesy of R. Weiss, 2005)



Acoustic detector

A second technique employs long baselines between test masses because the displacements for a given strain are proportionally larger. Two test masses (mirrors) are placed kilometers apart in a vacuum chamber and their displacement is measured by means of a laser interferometer in a configuration similar to a Michelson interferometer; the arms are Fabry-Perot cavities. A strain of 10^{-21} over a distance of 7 km gives a displacement of 2×10^{-18} m. For a typical laser wavelength of 10670 Å, a single pass fractional fringe shift for this strain is 7×10^{-12}. If the Fabry-Perot cavities have a finesse of several hundred, the effective path length is increased by the same factor. (Figure on next page courtesy of R. Weiss, 2005)



Representative detectors.

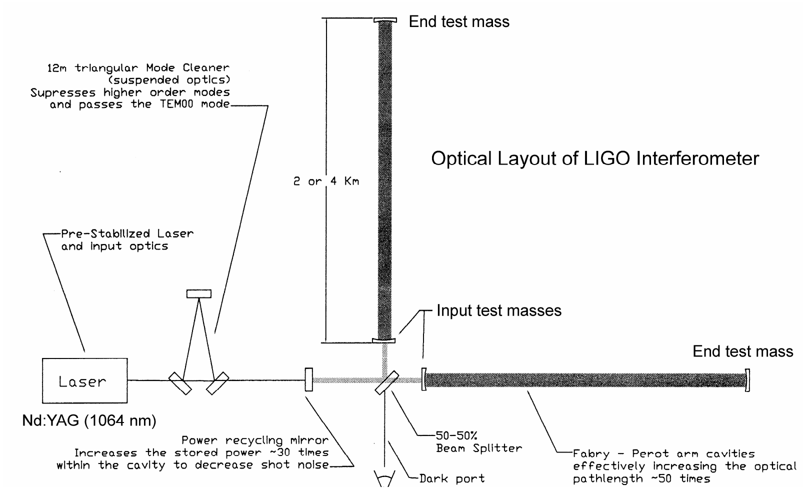
Technique	Detector	Frequency Range	Strain Sensitivity
Acoustic resonator (Weber bar)	Allegro(1)	890 – 930 Hz	10^{-22} @ 900 Hz, 1 Hz bandwidth
Laser interferometer	LIGO(2)	70 Hz – 1 kHz	See below
Spaceborne Laser Interferometer	LISA(3)	10^{-7} – 1 Hz	See below

(1) **Allegro** (**A** Louisiana **L**ow temperature **E**xperiment and **G**ravitational wave **O**bservatory), cryogenic mass (aluminum, 2.5 tons) detector with a superconducting inductive transducer and a SQID amplifier, located in Baton Rouge, LA. See http://gravity.phys.lsu.edu/.

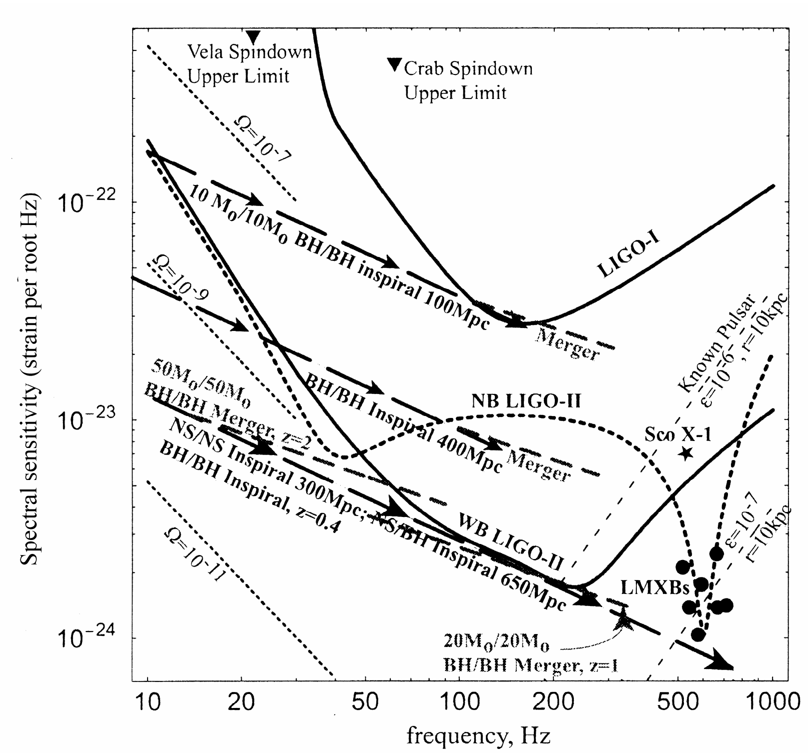
(2) **LIGO** (**L**aser **I**nterferometer **G**ravitational-Wave **O**bservatory), three Michelson interferometers, two located in Hanford, WA (with 2 and 7 km arms) and one in Livingston, LA (7 km arms). See http://www.ligo-wa.caltech.edu/ and http://www.ligo-la.caltech.edu/.

(3) **LISA** (**L**aser **I**nterferometer **S**pace **A**ntenna), (NASA/ESA), a laser interferometer orbiting the Sun at 1 AU, 20° behind the Earth proposed for 2009-2013; three spacecraft forming equilateral triangle, each side 5 x 10^6 km. See http://lisa.jpl.nasa.gov/.

Simplified optical layout of a LIGO interferometer. Servo loops ensure that the recombined light destructively interferes so that the dark port is kept dark. The gravitational wave signal is proportional to the force required to keep the recombined light in destructive interference. (Adapted from *New physics and astronomy with the new gravitational-wave observatories*, Hughes, S.A. *et al.*, aeXiv:astro-ph/0110379 v2 31 Oct 2001).

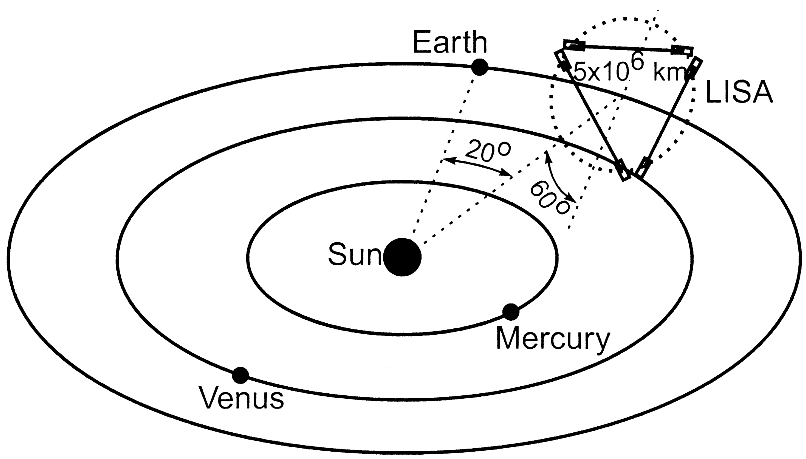


Comparisons of the root mean square noise spectral densities of LIGO detectors with spectral intensities of various sources *vs* gravitational wave frequency. The signal strength is defined in such a way that wherever a signal point or curve lies above the detector's noise curve, the signal, coming from a random direction on the sky and with random orientation, is detectable with a false alarm probability of less than 1 per cent. (Adapted from *An Overview of Gravitational-Wave Sources*, Cutler, C. and Thorne, K.S., arXiv:gr-qc/0207090 v1 30 Apr 2002).

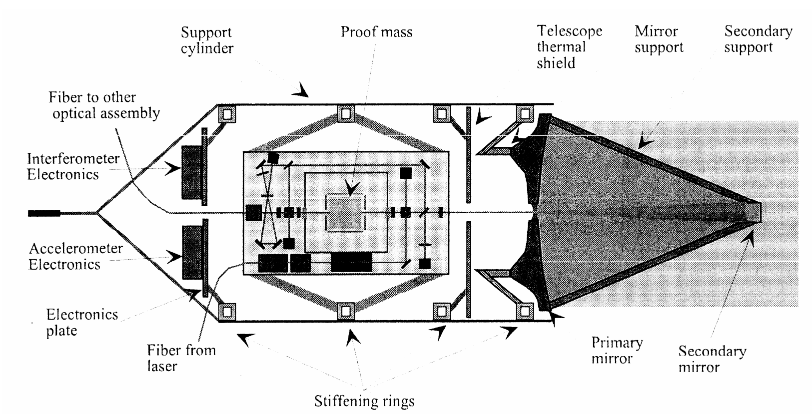


Ω - gravitational wave energy density (stochastic background) in a bandwidth equal to frequency in units of the closure density.
ε - gravitational ellipticity
r – source distance
BH – black hole
NS – neutron star
NB – narrow band
WB – wide band
LIGO-I – first-generation LIGO
LIGO-I I – second-generation LIGO
Assumptions – H_0 = 65 km s^{-1} Mpc^{-1}, Ω_M = 0.7, and Ω_Λ = 0.6.

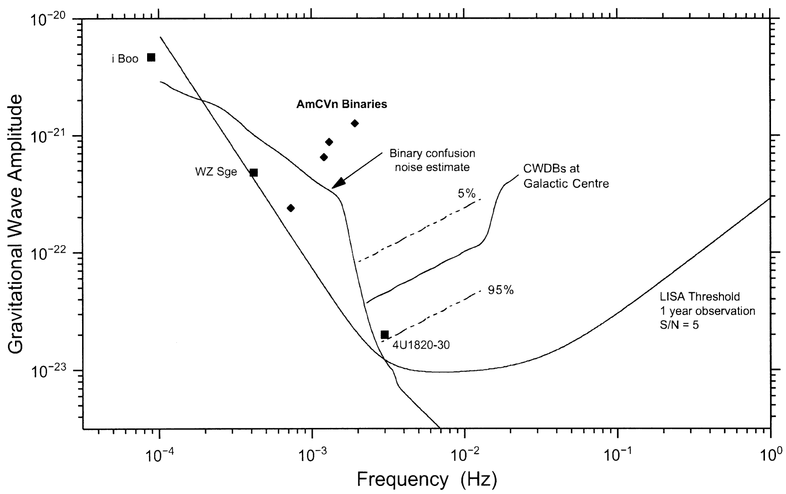
Orbital configuration of the LISA antennae (Hughes, S.A. *et al*., *op. cit*.).



Schematic of the LISA optical assembly (Hughes, S.A. *et al*., *op. cit*.).



LISA threshold sensitivity and signal levels and frequencies for a few known galactic sources; 1 year observation SNR = 5. (From the LISA Prephase A Report, 1995.)



Rainer Weiss of MIT provided useful comments for this section.

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Hughes, S.A. *et al*., *New physics and astronomy with the new gravitational-wave observatories*, Proceedings of the 2001 Snowmass Meeting; LIGO Report No. LIGO-P010029-00-D, ITP Report No. NSF-ITP-01-160, http://xxx.lanl.gov/PS_cache/astro-ph/pdf/0110/0110379.pdf.
Weiss, R. *Gravitational Radiation*, in *A Celebration of Physics at the Millennium*, Bederson, B., ed., Springer, American Physical Society, 1999.

Online list of detectors:

http://www.johnstonsarchive.net/relativity/gwdtable.html.

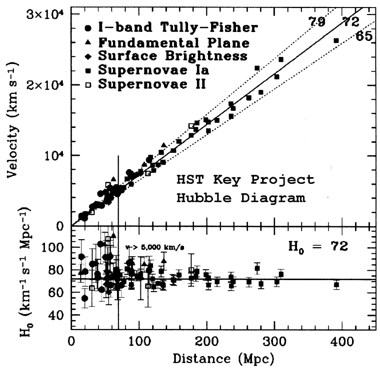
Cosmological parameters

After twenty years, we now have the first direct evidence that the Universe might be flat, but we also have definitive evidence that there is not enough matter, including dark matter, to make it so. We seem to be forced to accept the possibility that some weird form of dark energy is the dominant stuff in the Universe. – L.M. Krauss, 2002

The Friedmann-Lamaitre-Robertson-Walker model (see Chapter 10) incorporating inflation can be described by 16 cosmological parameters. The table below lists these parameters as of 2003 (from *Colloquium: Measuring and understanding the universe*, Freedman, W. and Turner, M.S., Rev. Mod. Phys., **75**, 1733, 2003). They represent Freedman and Turner's analysis of published data and are compared to the results from WMAP (Wilkinson Microwave Anisotropy Probe satellite).

Parameter	value	Description	WMAP
		Ten global parameters	
h	0.72 ± 0.07	present expansion rate[a]	$0.71^{+0.04}_{-0.03}$
q_0	-0.67 ± 0.25	deceleration parameter[b]	-0.66 ± 0.10
t_0	13 ± 1.5 Gyr	age of the universe[c]	13.7 ± 0.2 Gyr
T_0	2.725 ± 0.001 K	CMB temperature	
Ω_0	1.03 ± 0.03	density parameter[d]	1.02 ± 0.02
Ω_B	0.039 ± 0.008	baryon density	0.044 ± 0.004
Ω_{CDM}	0.29 ± 0.04	cold dark matter density	0.23 ± 0.04
Ω_ν	$0.001-0.05$	massive neutrino density	
Ω_X	0.67 ± 0.06	dark energy density	0.73 ± 0.04
w	-1 ± 0.2	dark energy equation of state	<-0.8 (95% cl)
		Six fluctuation parameters[e]	
$\sqrt{S}$	$5.6^{+1.5}_{-1.0}\times10^{-6}$	density perturbation amplitude[f]	
$\sqrt{T}$	$<\sqrt{S}$	gravity wave amplitude[g]	$T<0.9S$ (95% cl)
σ_8	0.9 ± 0.1	mass fluctuations on 8 Mpc[h]	0.84 ± 0.04
n	1.05 ± 0.09	scalar index[i]	0.93 ± 0.03
n_T		tensor index[j]	
$dn/d\ln k$	-0.02 ± 0.04	running of scalar index[k]	-0.03 ± 0.02

[a] $H_0 = 100h$ km s^{-1} Mpc^{-1}



[b] $q_0 \equiv -(d^2a/dt^2/a)_0/H_0^2 = \Omega_0/2 + (3/2)w_X\Omega_X$, where a is the cosmic scale factor, Ω_0 is the density parameter, $w_X \equiv p_X/\rho_X$ characterizes the pressure of the dark energy component, Ω_X is the dark energy density.

[c] $$t_0 = (1/H_0)\int_0^{\infty}[(\Omega_M)(1+z)^3 + (\Omega_X)(1+z)^{3(1+w)}]^{-1/s}(1+z)^{-1}\,dz,$$

$\Omega_M = \Omega_{CDM} + \Omega_B + \Omega_\nu$ is the total mass density parameter.

[d] $\Omega_0 = \rho_{tot}/\rho_{crit}$, where ρ_{tot} is the mass-energy density, $\rho_{crit} \equiv 3H_0^2/8\pi G$, the "critical density", and G is the universal gravitational constant.

[e] These parameters characterize deviations from homogeneity in the Universe.

[f] Contribution of density perturbations to the variance of the CMB quadrupole (with $T = 0$).

[g] Contribution of gravity waves to the variance of the CMB quadrupole (upper limit).

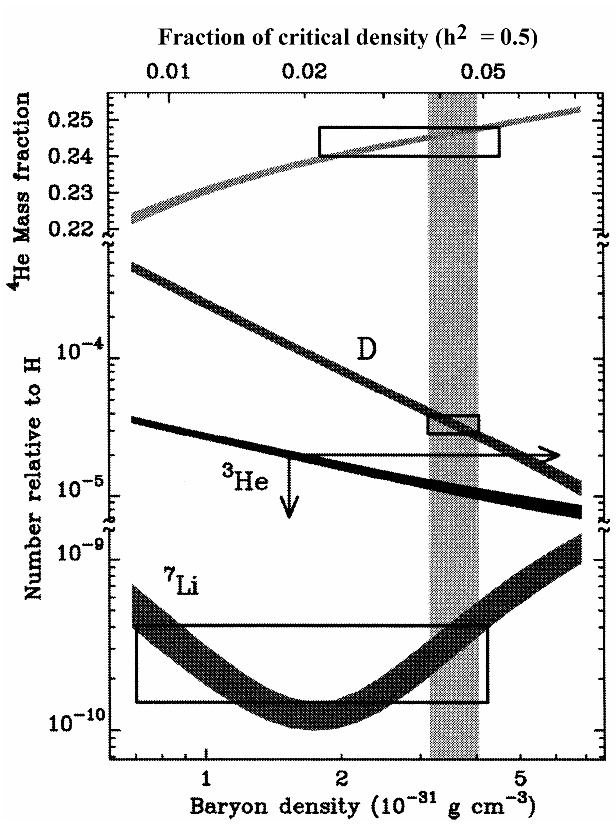
[h] The amplitude of fluctuations on a scale oh $8h^{-1}$ Mpc.

[i] Index of the power law ($P(k)$; k is the wave number) describing primordial density fluctuations. n = 1 corresponds to fluctuations in the gravitational potential that are the same on all scales.

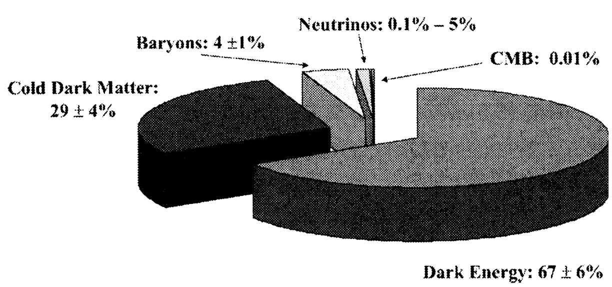
[j] Index for gravity wave perturbations to the CMB quadrupole .

[k] Deviation of the scalar perturbations from a power law.

The predicted abundance of the light elements *vs.* baryon density.
(Adapted from Freedman, W. and Turner, M.S , *op. cit.*)



The matter and energy in the present Universe.
(Adapted from Freedman, W. and Turner, M.S , *op. cit.*)



For an up-to-date listing of cosmological parameters see:
The Review of Particle Physics, http://pdg.lbl.gov/.

Cosmic Microwave Background (CMB)

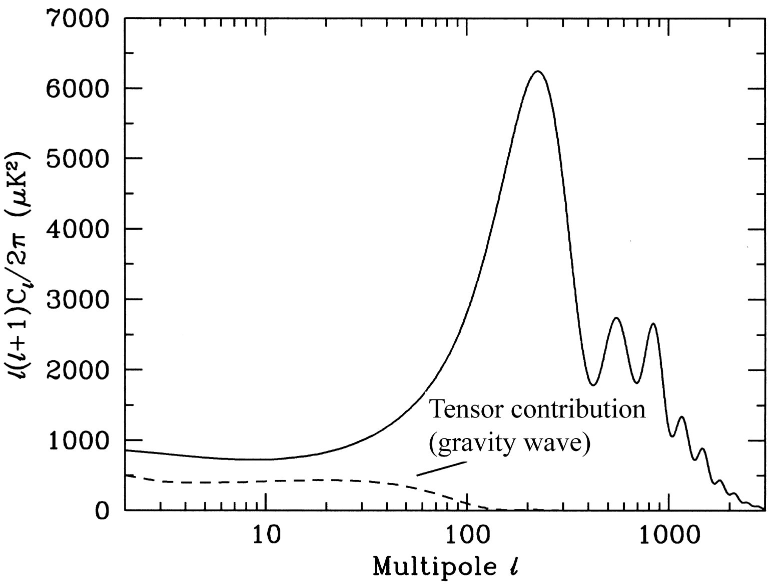
The cosmic microwave background is a 2.725 ± 0.001 K thermal spectrum of black body radiation that fills the universe. It has a peak frequency of 160.7 GHz which corresponds to a wavelength of 1.9 mm. The energy density of the CMB is $0.26038(T/2.725)^7$ eV cm^{-3} and the number density of CMB photons is $710.50(T/2.725)^3$ cm^{-3}. It is isotropic to roughly one part in 100,000 over a wide range of angular scales: the root mean square variations are only 18 μK. The anisotropies are usually expressed as a spherical harmonic expansion of the CMB sky:

$$T(\theta,\phi) = \sum_{lm} a_{lm} Y_{lm}(\theta,\phi)$$

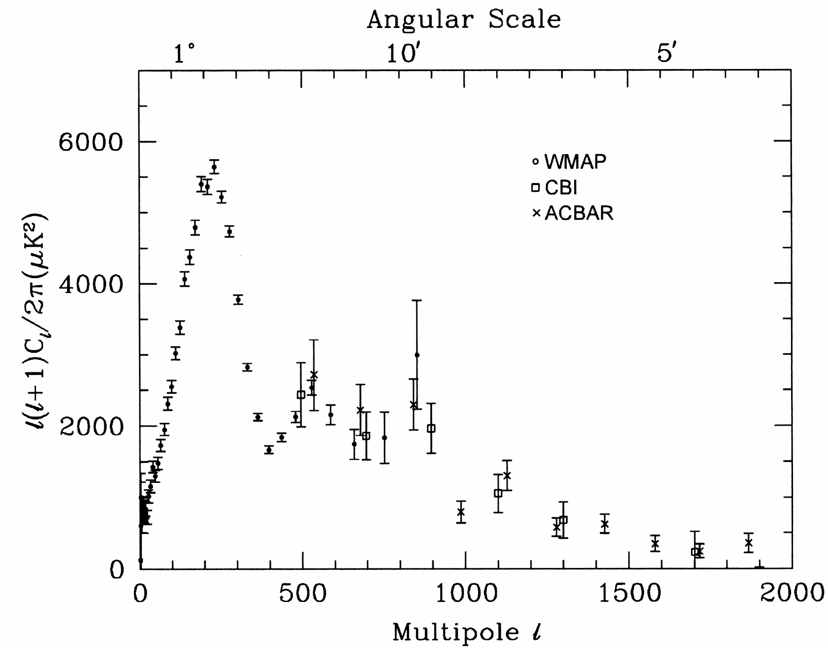
The power per unit ln l is $l\sum_m |a_{lm}|^2 / 4\pi$ with $l \sim 1/\theta$. The mean temperature of 2.725 K can be considered as the monopole component of CMB maps, a_{00}. The largest anisotropy is in the $l = 1$ (dipole) first spherical harmonic with an amplitude of 3.376 ± 0.017 mK. The higher multipole amplitudes are interpreted as the result of perturbations in the energy density of the early Universe. The power at each l is $(2l + 1)C_l/7\pi$, where

$$C_l \equiv \left\langle [a_{lm}]^2 \right\rangle .$$

A theoretical CMB anisotropy power spectrum, using a standard Cold Dark Matter plus Cosmological Constant model.



Power estimates from WMAP (Wilkinson Microwave Anisotropy Probe satellite), CBI (Cosmic Background Imager), and ACBAR (Arcminute Cosmology Bolometer Array Receiver).



The material for this section is based on the discussion by D. Scott and G.F. Smoot in S. Eidelman *et al*., Physics Letters **B592**, 1, 2007. See also *The Review of Particle Physics*, http://pdg.lbl.gov/.