Massachusetts Institute of Technology
Department of Physics

Physics 8.962 Spring 2002

Tensor Calculus, Part 2

1 Introduction

The first set of 8.962 notes, *Introduction to Tensor Calculus for General Relativity*, discussed tensors, gradients, and elementary integration. The current notes continue the discussion of tensor calculus with orthonormal bases and commutators (§2), parallel transport and geodesics (§3), and the Riemann curvature tensor (§4).

2 Orthonormal Bases, Tetrads, and Commutators

A vector basis is said to be orthonormal at point **X** if the dot product is given by the Minkowski metric at that point:

$$\{\vec{e}_{\hat{\mu}}\} \text{ is orthonormal if and only if } \vec{e}_{\hat{\mu}} \cdot \vec{e}_{\hat{\nu}} = \eta_{\mu\nu} \ . \tag{1}$$

(We have suppressed the implied subscript **X** for clarity.) Note that we will always place a hat over the index for any component of an orthonormal basis vector. The smoothness properties of a manifold imply that it is always possible to choose an orthonormal basis at any point in a manifold. One simply choose a basis that diagonalizes the metric g and furthermore reduces it to the normalized Minkowski form. Indeed, there are infinitely many orthonormal bases at **X** related to each other by Lorentz transformations. Orthonormal bases correspond to locally inertial frames.

For each basis of orthonormal vectors there is a corresponding basis of orthonormal one-forms related to the basis vectors by the usual duality condition:

$$\left\langle \tilde{e}^{\hat{\mu}}, \vec{e}_{\hat{\nu}} \right\rangle = \delta^{\mu}{}_{\nu} \ . \tag{2}$$

The existence of orthonormal bases at one point is very useful in providing a locally inertial frame in which to present the components of tensors measured by an observer at rest in that frame. Consider an observer with 4-velocity $\vec{V}$ at point $\mathbf{X}$. Since $\vec{V}\cdot\vec{V}=-1$, the observer's rest frame has timelike orthonormal basis vector $\vec{e}_{\hat{0}}=\vec{V}$. The observer has a set of orthonormal space axes given by a set of spatial unit vectors $\vec{e}_{\hat{i}}$. For a given $\vec{e}_{\hat{0}}$, there are of course many possible choices for the spatial axes that are related by spatial rotations. Each choice of spatial axes, when combined with the observer's 4-velocity, gives an orthonormal basis or tetrad. Thus, an observer carries along an orthonormal bases that we call the **observer's tetrad**. This basis is the natural one for splitting vectors, one-forms, and tensors into timelike and spacelike parts. We use the observer's tetrad to extract physical, measurable quantities from geometric, coordinate-free objects in general relativity.

For example, consider a particle with 4-momentum $\vec{P}$. The energy in the observer's instantaneous inertial local rest frame is $E=-\vec{V}\cdot\vec{P}=-\vec{e}_{\hat{0}}\cdot\vec{P}=\langle\tilde{e}^{\hat{0}},\vec{P}\rangle$. The observer can define a $(2,0)$ projection tensor

$$\mathsf{h}\equiv\mathsf{g}^{-1}+\vec{V}\otimes\vec{V} \tag{3}$$

with components (in any basis) $h^{\alpha\beta}=g^{\alpha\beta}+V^\alpha V^\beta$. This projection tensor is essentially the inverse metric on spatial hypersurfaces orthogonal to $\vec{V}$; the corresponding $(0,2)$ tensor is $h_{\mu\nu}=g_{\alpha\mu}g_{\beta\nu}h^{\alpha\beta}$. The reader can easily verify that $h_{\mu\nu}V^\mu=h_{\mu\nu}V^\nu=0$, hence in the observer's tetrad, $h^{\hat{\mu}\hat{\nu}}=h_{\hat{\mu}\hat{\nu}}=\mathbf{diag}(0,1,1,1)$. Then, the spatial momentum components follow from $P^{\hat{i}}=\langle\tilde{e}^{\hat{i}},\vec{P}\rangle=P_{\hat{i}}=\vec{e}_{\hat{i}}\cdot\vec{P}$. (Normally it is meaningless to equate components of one-forms and vectors since they cannot be equal in all bases. Here we are restricting ourselves to a single basis — the observer's tetrad — where it happens that spatial components of one-forms and vectors are equal.) Note that $P^{\hat{i}}\vec{e}_{\hat{i}}=\mathsf{h}(\mathsf{g}(\vec{P}\,))$: the spatial part of the momentum is extracted using h. Thus, in any basis, $P^\mu=EV^\mu+h^\mu{}_\nu P^\nu$ splits $\vec{P}$ into parts parallel and perpendicular to $\vec{V}$. (Note $h^\mu{}_\nu\equiv g_{\kappa\nu}h^{\mu\kappa}$.)

2.1 Tetrads

If one can define an orthonormal basis for the tangent space at any point in a manifold, then one can define a set of orthonormal bases for **every** point in the manifold. In this way, equation (1) applies everywhere. At all spacetime points, the dot product has been reduced to the Minkowski form: $g_{\hat{\mu}\hat{\nu}}=\eta_{\hat{\mu}\hat{\nu}}$. One then has an orthonormal basis, or tetrad, for all points of spacetime.

If spacetime is not flat, how can we reduce the metric at every point to the Minkowski form? Doesn't that require a globally flat, Minkowski spacetime? How can one have the Minkowski metric without having Minkowski spacetime?

The resolution of this paradox lies in the fact that the metric we introduced in a coordinate basis has at least three different roles, and only one of them is played by $\eta_{\hat{\mu}\hat{\nu}}$. First, the metric gives the dot product: $\vec{A}\cdot\vec{B}=g_{\mu\nu}A^\mu B^\nu=\eta_{\hat{\mu}\hat{\nu}}A^{\hat{\mu}}B^{\hat{\nu}}$. Both $g_{\mu\nu}$

and $\eta_{\hat{\mu}\hat{\nu}}$ fulfill this role. Second, the metric components in a coordinate basis give the connection through the well-known Christoffel formula involving the partial derivatives of the metric components. Obviously since $\eta_{\hat{\mu}\hat{\nu}}$ has zero derivatives, it cannot give the connection. Third, the metric in a coordinate basis gives spacetime length and time through $d\vec{x} = dx^\mu \vec{e}_\mu$. Combining this with the dot product gives the line element, $ds^2 = d\vec{x} \cdot d\vec{x} = g_{\mu\nu} dx^\mu dx^\nu$. This formula is true only in a coordinate basis!

Usually when we speak of "metric" we mean the metric in a coordinate basis, which relates coordinate differentials to the line element: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. An orthonormal basis, unless it is also a coordinate basis, does not have enough information to provide the line element (or the connection). To determine these, we must find a linear transformation from the orthonormal basis to a coordinate basis:

$$\vec{e}_\mu = E^{\hat{\mu}}{}_\mu \vec{e}_{\hat{\mu}} \ . \tag{4}$$

The coefficients $E^{\hat{\mu}}{}_\mu$ are called the **tetrad components**. Note that $\hat{\mu}$ labels the (tetrad) basis vector while μ labels the component in some coordinate system (which may have no relation at all to the orthonormal basis). For a given orthonormal basis, $E^{\hat{\mu}}{}_\mu$ may be regarded as (the components of) a set of 4 one-form fields, one one-form $\tilde{E}^{\hat{\mu}} = E^{\hat{\mu}}{}_\mu \tilde{e}^\mu$ for each value of $\hat{\mu}$. Note that the tetrad components are *not* the components of a (1,1) tensor because of the mixture of two different bases.

The tetrad may be inverted in the obvious way:

$$\vec{e}_{\hat{\mu}} = E^\mu{}_{\hat{\mu}} \vec{e}_\mu \quad \text{where} \quad E^\mu{}_{\hat{\mu}} E^{\hat{\mu}}{}_\nu = \delta^\mu{}_\nu \ . \tag{5}$$

The dual basis one-forms are related by the tetrad and its inverse as for any change of basis: $\tilde{e}^\mu = E^\mu{}_{\hat{\mu}} \tilde{e}^{\hat{\mu}}$, $\tilde{e}^{\hat{\mu}} = E^{\hat{\mu}}{}_\mu \tilde{e}^\mu$,

The metric components in the coordinate basis follow from the tetrad components:

$$g_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu = \eta_{\hat{\mu}\hat{\nu}} E^{\hat{\mu}}{}_\mu E^{\hat{\nu}}{}_\nu \tag{6}$$

or $g = E^T \eta E$ in matrix notation. Sometimes the tetrad is called the "square root of the metric." Equation (6) is the key result allowing us to use orthonormal bases in curved spacetime.

To discuss the curvature of a manifold we first need a connection relating nearby points in the manifold. If there exists any basis (orthonormal or not) such that $\langle \tilde{e}^\lambda, \widetilde{\nabla} \vec{e}_\mu \rangle \equiv \Gamma^\lambda{}_{\mu\nu} \tilde{e}^\nu = 0$ everywhere, then the manifold is indeed flat. However, the converse is not true: if the basis vectors rotate from one point to another even in a flat space (e.g. the polar coordinate basis in the plane) the connection will not vanish. Thus we will need to compute the connection and later look for additional quantities that give an invariant (basis-free) meaning to curvature. First we examine a more primitive object related to the gradient of vector fields, the commutator.

2.2 Commutators

The difference between an orthonormal basis and a coordinate basis arises immediately when one considers the commutator of two vector fields, which is a vector that may symbolically be defined by

$$[\vec{A},\vec{B}\,] \equiv \nabla_A\nabla_B - \nabla_B\nabla_A \tag{7}$$

where ∇_A is the directional derivative ($\nabla_A = A^\mu\partial_\mu$ in a coordinate basis). Equation (7) introduces a new notation and new concept of a vector since the right-hand side consists solely of differential operators with no arrows! To interpret this, we rewrite the right-hand side in a coordinate basis using, e.g., $\nabla_A\nabla_B f = A^\mu\partial_\mu(B^\nu\partial_\nu f)$ (where f is any twice-differentiable scalar field):

$$[\vec{A},\vec{B}\,] = \left(A^\mu\frac{\partial B^\nu}{\partial x^\mu} - B^\mu\frac{\partial A^\nu}{\partial x^\mu}\right)\frac{\partial}{\partial x^\nu}\ . \tag{8}$$

This is equivalent to a vector because $\{\partial/\partial x^\nu\}$ provide a coordinate basis for vectors in the formulation of differential geometry introduced by Cartan. Given our heuristic approach to vectors as objects with magnitude and direction, it seems strange to treat a partial derivative as a vector. However, Cartan showed that directional derivatives form a vector space isomorphic to the tangent space of a manifold. Following him, differential geometry experts replace our coordinate basis vectors $\vec{e}_\mu$ by $\partial/\partial x^\mu$. (MTW introduce this approach in Chapter 8. On p. 203, they write $\vec{e}_\alpha = \partial\mathcal{P}/\partial x^\alpha$ where $\mathcal{P}$ refers to a point in the manifold, as a way to indicate the association of the tangent vector and directional derivative.) With this choice, vectors become differential operators (e.g. $\vec{A} = A^\mu\partial_\mu$) and thus the commutator of two vector fields involves derivatives. However, we need not follow the Cartan notation. It is enough for us to define the commutator of two vectors by its components in a coordinate basis,

$$[\vec{A},\vec{B}\,] = (A^\mu\partial_\mu B^\nu - B^\mu\partial_\mu A^\nu)\vec{e}_\nu \quad \text{in a coordinate basis,} \tag{9}$$

where the partial derivative operators act only on B^ν and A^ν but not on $\vec{e}_\nu$.

Equation (9) implies

$$[\vec{A},\vec{B}\,] = \nabla_A\vec{B} - \nabla_B\vec{A} + T^\mu{}_{\alpha\beta}A^\alpha B^\beta\vec{e}_\mu\ , \tag{10}$$

where $T^\mu{}_{\alpha\beta} \equiv \Gamma^\mu{}_{\alpha\beta} - \Gamma^\mu{}_{\beta\alpha}$ in a coordinate basis is a quantity called the torsion tensor. The reader may easily show that the torsion tensor also follows from the commutator of covariant derivatives applied to any twice-differentiable scalar field,

$$(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)f = T^\mu{}_{\alpha\beta}\nabla_\mu f \tag{11}$$

This equation shows that the torsion is a tensor even though the connection is not. The torsion vanishes by assumption in general relativity. This is a statement of physics, not mathematics. Other gravity theories allow for torsion to incorporate possible new physical effects beyond Einstein gravity.

The basis vector fields $\vec{e}_\mu(x)$ are vector fields, so let us examine their commutators. From equation (9) or (10), in an coordinate basis, the commutators vanish identically (even if the torsion does not vanish):

$$[\vec{e}_\mu, \vec{e}_\nu] = 0 \quad \text{in a coordinate basis} \ . \tag{12}$$

The vanishing of the commutators occurs because the coordinate basis vectors are dual to an integrable basis of one-forms: $\tilde{e}^\mu = \widetilde{\nabla} x^\mu$ for a set of 4 scalar fields x^μ. It may be shown that this integrability condition (i.e. that the basis one-forms may be integrated to give functions) is equivalent to equation (12) (see Wald 1984, problem 5 of Chapter 2).

Now let us examine the commutator for an orthonormal basis. We use equation (9) by expressing the tetrad components in a coordinate basis using equation (5). The result is

$$[\vec{e}_{\hat\mu}, \vec{e}_{\hat\nu}] = \partial_{\hat\mu}\vec{e}_{\hat\nu} - \partial_{\hat\nu}\vec{e}_{\hat\mu} \equiv \omega^{\hat\alpha}{}_{\hat\mu\hat\nu}\vec{e}_{\hat\alpha} \ , \tag{13}$$

where $\partial_{\hat\mu} \equiv E^\mu{}_{\hat\mu}\partial_\mu$. Equation (13) defines the **commutator basis coefficients** $\omega^{\hat\alpha}{}_{\hat\mu\hat\nu}$ (cf. MTW eq. 8.14). Using equations (5), (12), and (13), one may show

$$\omega^{\hat\alpha}{}_{\hat\mu\hat\nu} = E^{\hat\alpha}{}_\alpha \left(\nabla_{\hat\mu}E^\alpha{}_{\hat\nu} - \nabla_{\hat\nu}E^\alpha{}_{\hat\mu}\right) = E^\mu{}_{\hat\mu}E^\nu{}_{\hat\nu}\left(\partial_\mu E^{\hat\alpha}{}_\nu - \partial_\nu E^{\hat\alpha}{}_\mu\right) \ . \tag{14}$$

In general the commutator basis coefficients do not vanish. Despite the appearance of a second (coordinate) basis, the commutator basis coefficients are independent of any other basis besides the orthonormal one. The coordinate basis is introduced solely for the convenience of partial differentiation with respect to the coordinates.

The commutator basis coefficients carry information about how the tetrad rotates as one moves to nearby points in the manifold. It is useful practice to derive them for the orthonormal basis $\{\vec{e}_{\hat r}, \vec{e}_{\hat\theta}\}$ in the Euclidean plane.

2.3 Connection for an orthonormal basis

The connection for the basis $\{\vec{e}_{\hat\mu}\}$ is defined by

$$\partial_{\hat\nu}\vec{e}_{\hat\mu} \equiv \Gamma^{\hat\alpha}{}_{\hat\mu\hat\nu}\vec{e}_{\hat\alpha} \ . \tag{15}$$

(The placement of the lower subscripts on the connection agrees with MTW but is reversed compared with Wald and Carroll.) From the local flatness theorem (metric compatibility with covariant derivative) discussed in the first set of notes,

$$\nabla_{\hat\alpha}g_{\hat\mu\hat\nu} = E^\alpha{}_{\hat\alpha}\partial_\alpha g_{\hat\mu\hat\nu} - \Gamma^{\hat\beta}{}_{\hat\mu\hat\alpha}g_{\hat\beta\hat\nu} - \Gamma^{\hat\beta}{}_{\hat\nu\hat\alpha}g_{\hat\mu\hat\beta} = 0 \ . \tag{16}$$

In an orthonormal basis, $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$ is constant so its derivatives vanish. We conclude that, in an orthonormal basis, the connection is antisymmetric on its first two indices:

$$\Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}} = -\Gamma_{\hat{\nu}\hat{\mu}\hat{\alpha}} \ , \quad \Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}} \equiv g_{\hat{\mu}\hat{\beta}}\Gamma^{\hat{\beta}}{}_{\hat{\nu}\hat{\alpha}} = \eta_{\hat{\mu}\hat{\beta}}\Gamma^{\hat{\beta}}{}_{\hat{\nu}\hat{\alpha}} \ . \tag{17}$$

In an orthonormal basis, the connection is *not*, in general, symmetric on its last two indices. (That is true only in a coordinate basis.)

Another equation for the connection coefficients comes from combining equations (13) with equation (15):

$$\omega_{\hat{\alpha}\hat{\mu}\hat{\nu}} = -\Gamma_{\hat{\alpha}\hat{\mu}\hat{\nu}} + \Gamma_{\hat{\alpha}\hat{\nu}\hat{\mu}} \ , \quad \omega_{\hat{\alpha}\hat{\mu}\hat{\nu}} \equiv g_{\hat{\alpha}\hat{\beta}}\omega^{\hat{\beta}}{}_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\alpha}\hat{\beta}}\omega^{\hat{\beta}}{}_{\hat{\mu}\hat{\nu}} \ . \tag{18}$$

Combining these last two equations yields

$$\Gamma_{\hat{\alpha}\hat{\mu}\hat{\nu}} = \frac{1}{2}\left(\omega_{\hat{\mu}\hat{\alpha}\hat{\nu}} + \omega_{\hat{\nu}\hat{\alpha}\hat{\mu}} - \omega_{\hat{\alpha}\hat{\mu}\hat{\nu}}\right) \quad \text{in an orthonormal basis.} \tag{19}$$

The connection coefficients in an orthonormal basis are also called Ricci rotation coefficients (Wald) or the spin connection (Carroll).

It is straightforward to generalize the results of this section to general bases that are neither orthonormal nor coordinate. The commutator basis coefficients are defined as in equation (12). Dropping the carets on the indices, the general connection is (MTW eq. 8.24b)

$$\Gamma_{\alpha\mu\nu} \equiv g_{\alpha\beta}\Gamma^{\beta}{}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} + \omega_{\mu\alpha\nu} + \omega_{\nu\alpha\mu} - \omega_{\alpha\mu\nu}\right) \quad \text{in any basis.} \tag{20}$$

The results for coordinate bases (where $\omega_{\alpha\mu\nu} = 0$) and for orthonormal bases (where $\partial_\alpha g_{\mu\nu} = 0$) follow as special cases.